SOME MOTIVATION, SOME RAMBLING

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The following is an excerpt from the introductory part of my PhD thesis. It is written in somewhat increasing levels of abstraction and difficulty, although most parts restart a bit gentler. This way, anyone should, hopefully, gain something from these ramblings.

(Algebraic) geometry. What actually is *geometry*? A short, rough answer, according to ChatGPT at least, could be

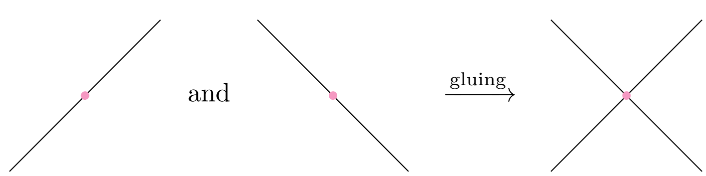
> 'Geometry is the branch of mathematics that studies shapes, their properties, and their relationships in space. It deals with concepts such as points, lines, angles, surfaces, and solids.'

In short, the study of *geometric objects*. Of course, this raises an obvious follow-up question: what is a geometric object?

When posing this question to my sister, she thought of points, lines, triangles, squares or cubes; objects typical in (planar) Euclidean geometry, which is how most people first encounter geometry in school. These objects have the benefit of being easy to visualise.

Of course, there is more to life than Euclidean geometry. Leaving the axioms of flat Euclidean geometry behind, other types of interesting phenomena appear; space can be curved. For example, to describe the theory of gravity in Einstein's general relativity, physicists make use of four-dimensional curved spaces to model the geometry of space-time (three space dimensions and one time dimension). These types of four-dimensional spaces are not easily visualised, but that does not make them any less real. They describe how nature works. A more down-to-earth example: the shortest path between two points on a map of the Earth is not a straight line precisely because the Earth is curved.

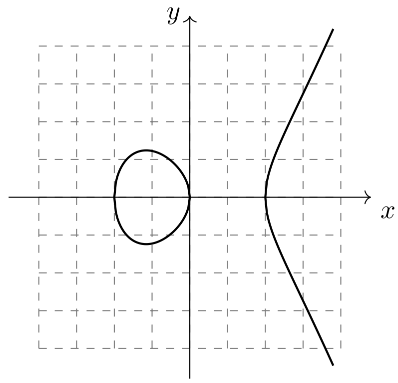
In *algebraic geometry* one studies geometric objects that are given by solutions to polynomial equations (locally at least). These solution sets are known as *affine varieties*. Using these as basic building blocks we can construct more general types of varieties by 'gluing' together affine ones. For example, gluing the two lines along a point on each of them gives a shape in the form of a cross.



As an example of an affine variety, let $f(x, y)$ be a polynomial in two variables. By looking at its zero set we can associate to it a curve in a plane. That is, we look at the set

$$\{pt \text{ in plane} \mid f(pt) = 0\}.$$

Taking the polynomial to be $y^2 - x^3 + x$ and looking at points in the real plane $\mathbb{R}^2$ we obtain the following zero set



Curves of the above type are known as elliptic curves. They admit a special structure as they have an addition on their set of points, an algebraic operation similar to the addition of integers. However, we need not only restrict to real solutions. We could exchange the field $\mathbb{R}$ by any finite field, an algebraic structure similar to the real numbers but with only a finite number of elements. The curve would then consist of a finite set of points, as the plane over a finite field only consists of a finite number of points; not what one would normally think of as a curve. Looking at elliptic curves over finite fields might seem silly, but the extra structure of an addition on their points has very important real world applications through elliptic curve cryptography.

The beauty of algebraic geometry is that we can describe these geometric objects purely algebraically. Namely, to a plane curve we can associate a *commutative* ring by looking at all the 'regular functions' on the curve. A ring is an algebraic structure consisting of elements that can be added and multiplied, just as we can do with integers or real numbers. Concretely, this ring of regular functions can be given as the quotient ring

$$\mathbb{R}[x, y]/(f)$$

of the real polynomial ring in two variables quotiented by the ideal generated by f, i.e. the subset of the polynomial ring containing all the multiples of f. It is commutative as the two variables x and y commute with each other, i.e. the order in which they are written does not matter

$$xy = yx.$$

More generally, to any commutative ring we can associate a geometric object, known as an *affine scheme*, and conversely to any affine scheme we can associate a commutative ring. This leads to a beautiful dictionary between geometry and commutative algebra

$$\{\text{affine schemes}\} \underset{\Gamma(-)}{\overset{\mathrm{Spec}(-)}{\rightleftarrows}} \{\text{commutative rings}\}.$$

The arrow Spec is the analogue of looking at the zero set of polynomials above, whilst the arrow Γ corresponds to considering the ring of regular functions. Just as with affine varieties, these affine schemes can be glued together to give more general geometric objects known simply as *schemes*.

Noncommutative (algebraic) geometry. Not all rings are commutative, and in fact many rings that one can naturally consider are *noncommutative*. For example, for those who remember, matrix multiplication is noncommutative:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So how geometric are noncommutative rings then? Noncommutative algebraic geometry tries to answer this question, in some sense. As alluded to above, in traditional algebraic geometry the focus is on studying the geometric properties of spaces which are governed by commutative structures. Noncommutative algebraic geometry on the other hand can be seen as both the geometric study of abstract noncommutative structures and as the application of these abstract structures to the study of geometric spaces.

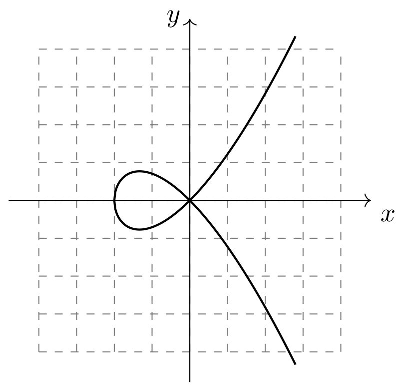
Noncommutative geometry does not directly yield some ‘noncommutative space’ that can be easily visualised. Rather it gives us a collection of objects that behave as if they were interacting with some would-be space. For this reason one may wonder how geometric this really is. However, my point of view is that this is irrelevant: they lead to interesting objects of study, whether they can be easily visualised or not; they still interact with us in meaningful ways. I like the following metaphor/analogy:

> We cannot see a black hole as no light escapes its event horizon. Thus we certainly cannot completely visualise it. However, of course, this does not mean it is not there, that it cannot interact with us through different means in meaningful ways (get close and you will find out).

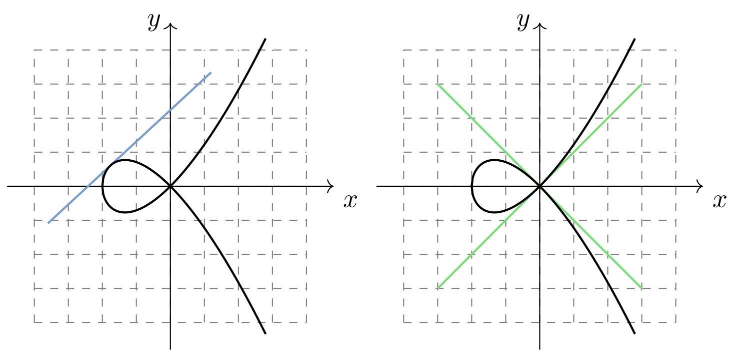
Similarly, we may not be able to see or visualise noncommutative spaces in the same way as we can the more common ‘commutative spaces’. This does, however, not mean that they cannot interact with us in meaningful ways, and through that interaction give us a better understanding of the mathematical landscape.

One way of thinking of a noncommutative space is as some structure that behaves as if it were interacting with some would-be space. To any variety, or more generally any scheme, we can associate objects called *(quasi-coherent) sheaves* that ‘live’ over it. In a sense that can be made precise, understanding this collection of sheaves is enough to recover the geometry of the variety over which they live. Together they form what is called a *category*, a collection of objects and arrows between the objects dictating how they relate to each other. Thus, this category contains the geometric information of the variety. We can therefore posit that a noncommutative space is a ‘nice enough’ category that behaves as if it was obtained from some would-be space. Depending on the specific context, the exact definition of ‘nice enough’ will change.

Resolutions of singularities. Not all varieties are created equal. Some are better behaved than others. In mathematics one of the big guiding questions in any branch is ‘can we classify all objects up to some notion of sameness?’. A concrete instance of this in algebraic geometry is ‘can we classify all varieties up to isomorphism?’ However, if we want to classify all varieties, which is a difficult task, it might be worthwhile to start classifying those varieties which are at least a bit better behaved and ask ourselves the question how much the 'good' and 'bad' ones differ. For example, let us look at the curve defined by the polynomial $y^2 - x^3 - x^2$. It is known as a nodal cubic.

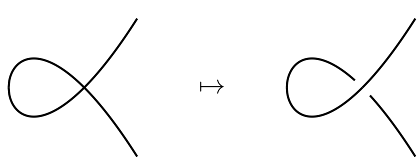


Clearly the origin looks different from the other points; the curve intersects itself there. For a point distinct from the origin, we can draw a unique tangent line at that point, a unique linear approximation of the curve. However, at the origin we cannot. There are two different lines approximating the curve at the origin.

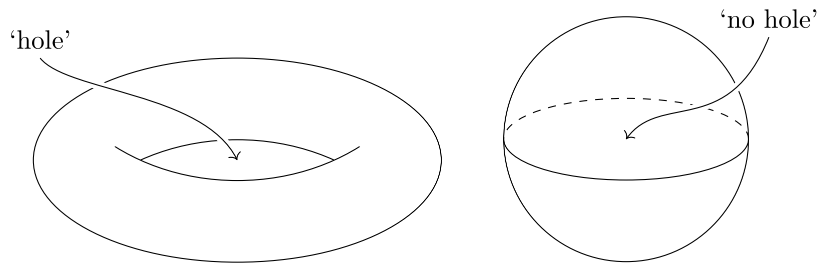


This observation reflects the fact that the origin is a *singular* point of the curve whilst all the other points are *smooth*. Smooth points are better behaved, and therefore smooth curves, where all points are smooth, or more generally smooth varieties, are desirable.

There is a procedure, a type of surgery we can do, known as *a resolution of singularities*, by which we can nicely approximate a singular variety by a smooth variety, thereby getting rid of the 'badly behaved' points. For a curve this approximation would mean, amongst other things, that the singular and smooth curve have to be the same except for a finite number of points. For example, we can resolve the nodal cubic above by 'blowing up the origin', essentially this separates the two tangent lines at the origin. Almost all points stay the same, except that the origin gets replaced by two points.



Noncommutative resolutions. Often in order to better understand a mathematical object it is beneficial to associate *invariants* to it. Roughly speaking this is a characteristic of the object that remains unchanged under certain operations or transformations. For example, the colour of an object does not change when we move it through space. These invariants could themselves be other (hopefully simpler) mathematical objects, but could also be something as simple as a number. One tool kit to make such invariants is *homological algebra*. It allows us to associate algebraic objects as invariants through defining what is called *(co)homology*. For example, through (co)homology we can detect 'holes' in surfaces. A torus (in less mathematical lingo, a doughnut) has a hole in it, whilst a sphere does not. (Co)homology can detect this, it will give a one for the torus and a zero for the sphere.



Calculating invariants is where noncommutative geometry can be of use. The idea of using noncommutative structures to resolve singular varieties and compute relevant invariants appeared quite early on in physics[1], see e.g. [BL01]. Around the same time noncommutative resolutions started to appear in pure mathematics, precursors would be [KV00, Lun01, BKR01]. The most elegant application, in my opinion, of noncommutative resolutions is the proof of the Bondal–Orlov conjecture in dimension three by Van den Bergh [VdB04]. To show that two commutative resolutions are the same, he related them both to noncommutative resolutions of which it is easier to show that they are the same.

In homological algebra, one constructs algebraic invariants by considering sequences of objects. For example, let R be a ring and M be an object with which the ring interacts in a suitable way (an R-module). To associate invariants to M one takes a nice resolution of this object, i.e. a sequence of arrows

$$\cdots \to F_n \to \cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

with the F_i of a specific form ('projective'), that 'fit together' nicely; in mathematical lingo: the sequence is exact. It would be nice if this sequence could be taken to be finite, that is $F_n = 0$ for large enough n. After all, finite is usually easier than infinite. Unfortunately, this is not always possible, and this is precisely related to the idea of being smooth. For a 'nice enough' commutative ring we can find for any such object M a finite resolution if and only if its associated affine scheme is smooth.

However, can we add extra objects to the category, so that it is always possible to construct such a finite sequence? This is exactly what a *categorical resolution* is supposed to do; to enlarge a category, in a suitable way, and make it smooth.

[1]We should mention that another form of noncommutative geometry, à la Connes, has also found inspiration/applications in physics and arrived earlier. As this uses operator algebras, it has a different flavour and we do not consider it here.

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